THE DEVELOPMENT OF WAVES GENERATED BY THE VIBRATIONS OF A STRIP

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The development of the wave process excited by the vibrations of a strip immersed in a fluid is investigated; the form of the undamped waves which are generated is found, the energy of these waves is calculated and compared to the energy of the vibrating strip.

The analogous problem for the case of steady motion has been examined by Alblas [1].

1. Let the fluid fill the part of space $y > 0$, $z < 0$. From $z = 0$ to $z = -h$ in the plane $y = 0$, there is located a strip which, beginning at the time $t = 0$, performs vibrations according to the law $y = a \exp[i(kx - \omega t)]$. The continuation of this strip from $z = -h$ to $z = -\infty$ is a solid immovable wall. At the initial time $t = 0$ the fluid is at rest and its free surface is horizontal.

We shall set ourselves the problem of finding the form of the free surface of the fluid at any time $t > 0$. The velocity potential $\phi(x,y,z,t)$ of the unknown wave motion must satisfy the equations

$$\Delta\varphi = 0 \tag{1.1}$$

$$\frac{\partial^2\varphi}{\partial t^2} + g\frac{\partial\varphi}{\partial z} = 0 \quad \text{at} \quad z = 0, \qquad \left(\frac{\partial\varphi}{\partial y}\right)_{y=0} = \begin{cases} 0 & (z < -h) \\ ai\omega e^{i(kx-\omega t)} & (z > -h) \end{cases} \tag{1.2}$$

$$\varphi(x, y, 0, 0) = 0, \qquad \frac{\partial\varphi(x, y, 0, 0)}{\partial t} = 0 \tag{1.3}$$

We shall seek a function $\phi_1(x, y, z, t)$ which satisfies conditions (1.1) and (1.2); for that purpose we shall represent it in the form

$$\varphi_1(x, y, z, t) = \varkappa(y, z)\, e^{i(kx-\omega t)} \tag{1.4}$$

Then $\kappa(y, z)$ must satisfy the following equations:

$$\frac{\partial^2 \varkappa}{\partial y^2} + \frac{\partial^2 \varkappa}{\partial z^2} - k^2 \varkappa = 0 \qquad (1.5)$$

$$\sigma\varkappa - \frac{\partial \varkappa}{\partial z} = 0 \quad \text{at} \quad z = 0 \quad (\sigma = \omega^2 g^{-1}) \qquad (1.6)$$

$$\left(\frac{\partial \varkappa}{\partial y}\right)_{y=0} = \begin{cases} 0 & (z < -h) \\ ai\omega & (z > -h) \end{cases} \qquad (1.7)$$

Representing $\kappa(y, z)$ in the form

$$\varkappa(y, z) = ce^{[iy\sqrt{\sigma^2 - k^2} + \sigma z]} + \int_0^\infty A(\mu) e^{-y\sqrt{\mu^2 + k^2}} (\mu \cos \mu z + \sigma \sin \mu z)\, d\mu \quad (\sigma > k) \qquad (1.8)$$

we see that $\kappa(y, z)$ with arbitrary c and $A(\mu)$ satisfies Equations (1.5) and (1.6).

Satisfying condition (1.7), we have

$$i\sqrt{\sigma^2 - k^2} ce^{\sigma z} - \int_0^\infty A(\mu) \sqrt{\mu^2 + k^2} (\mu \cos \mu z + \sigma \sin \mu z)\, d\mu = \begin{cases} 0 & (z < -h) \\ ai\omega & (z > -h) \end{cases} \qquad (1.9)$$

Making use of the solution of this equation given in [2], we obtain

$$c = \frac{2a\omega}{\sqrt{\sigma^2 - k^2}} (1 - e^{-\sigma h}), \qquad A(\mu) = \frac{2ai\omega\, [\sigma(1 - \cos \mu h) - \mu \sin \mu h]}{\pi\mu \sqrt{\mu^2 + k^2}\, (\mu^2 + \sigma^2)} \qquad (1.10)$$

Therefore

$$\zeta_1 = \frac{1}{g} \left(\frac{\partial \varphi_1}{\partial t}\right)_{z=0} = -\frac{i\omega}{g} e^{i(kx - \omega t)} \left\{ ce^{iy\sqrt{\sigma^2 - k^2}} + \int_0^\infty \mu A(\mu) e^{-y\sqrt{\mu^2 + k^2}} d\mu \right\} \qquad (1.11)$$

where $A(\mu)$ and c are given by Formulas (1.10). We shall represent the velocity potential $\phi(x, y, z, t)$ in the form of a sum of three terms:

$$\varphi(x, y, z, t) = \varphi_1(x, y, z, t) + \varphi_2(x, y, z, t) + \varphi_3(x, y, z, t) \qquad (1.12)$$

Here

$$\varphi_2 = \int_0^\infty B(m) \cos my \exp\{i[kx - \vartheta(m) t] + z\sqrt{k^2 + m^2}\}\, dm \qquad (1.13)$$

$$\varphi_3 = \int_0^\infty D(m) \cos my \exp\{i[kx + \vartheta(m) t] + z\sqrt{k^2 + m^2}\}\, dm \qquad (1.14)$$

$$\vartheta(m) = \sqrt[4]{g^2 (k^2 + m^2)}$$

It is obvious that ϕ_2 and ϕ_3 with arbitrary $B(m)$ and $D(m)$ satisfy condition (1.1), the first equation of condition (1.2) and also the second equation of (1.2) with the right-hand side equal to zero. Satisfying conditions (1.3), we arrive at the following equations for determining $B(m)$ and $D(m)$:

$$\int_0^\infty [B(m)+D(m)]\cos my\,dm = -\varkappa(y,0)$$

$$\int_0^\infty [B(m)-D(m)]\,\vartheta(m)\cos my\,dm = -\omega\varkappa(y,0)$$

Hence

$$B(m) = -\frac{\omega+\vartheta(m)}{\pi\vartheta(m)}(J_1+J_2), \qquad D(m) = \frac{\omega-\vartheta(m)}{\pi\vartheta(m)}(J_1+J_2) \qquad (1.15)$$

where

$$J_1 = c\int_0^\infty e^{i\nu\sqrt{\sigma^2-k^2}}\cos my\,dy, \qquad J_2 = \int_0^\infty \cos my\,dy\int_0^\infty \mu A(\mu)\,e^{-\nu\sqrt{\mu^2+k^2}}\,d\mu$$

Performing the calculations, we find

$$B(m) = B_1(m)+B_2(m)$$

where

$$B_1(m) = -\frac{c}{2}\,\frac{\omega+\vartheta(m)}{\vartheta(m)}\left[\delta(q_1)+\frac{i}{\pi}P(q_1^{-1})+\delta(q_2)+\frac{i}{\pi}P(q_2^{-1})\right] \qquad (1.16)$$

$$q_1 = m+\sqrt{\sigma^2-k^2}, \qquad q_2 = -m+\sqrt{\sigma^2-k^2}$$

$$B_2(m) = -\frac{\omega+\vartheta(m)}{\vartheta(m)}\int_0^\infty \frac{\mu A(\mu)\sqrt{\mu^2+k^2}}{m^2+\mu^2+k^2}\,d\mu \qquad (1.17)$$

Here δ is the Dirac delta-function and P is the symbol for the principal value of the integral. Now, from (1.13) we obtain

$$\varphi_2(x,y,z,t) = \varphi_{21}(x,y,z,t)+\varphi_{22}(x,y,z,t)$$

where

$$\varphi_{21} = \int_0^\infty B_1(m)\cos my\exp\{i[kx-\vartheta(m)t]+z\sqrt{k^2+m^2}\}\,dm$$

$$\varphi_{22} = \int_0^\infty B_2(m)\cos my\exp\{i[kx-\vartheta(m)t]+z\sqrt{k^2+m^2}\}\,dm$$

Hence

$$\zeta_2 = \frac{1}{g}\left(\frac{\partial\varphi_2}{\partial t}\right)_{z=0} = \zeta_{21} + \zeta_{22} \tag{1.18}$$

where

$$\zeta_{21} = \frac{ic}{2g} e^{ikx}\Big\{2\omega\cos\sqrt{\sigma^2 - k^2}\,y e^{-i\omega t} + \tag{1.19}$$

$$+ \frac{i\sqrt{gk(\sigma^2 - k^2)}}{\pi k} P\int_0^\infty \frac{\omega_1 + \sqrt[4]{1+n^2}}{\sigma_1^2 - n^2 - 1}\left[e^{kyM_1(n)} + e^{kyM_2(n)}\right] dn\Big\}$$

$$\zeta_{22} = \frac{ik\sqrt{gk}}{2g} e^{ikx}\int_0^\infty \left(\omega_1 + \sqrt[4]{1+n^2}\right)\int_0^\infty \frac{\mu A(\mu)\sqrt{\mu^2 + k^2}}{k^2 n^2 + \mu^2 + k^2} d\mu \left[e^{kyM_1(n)} + e^{kyM_2(n)}\right] dn \tag{1.20}$$

$$M_1(n) = i\left(n - \nu\sqrt[4]{1+n^2}\right), \qquad M_2(n) = -i\left(n + \nu\sqrt[4]{1+n^2}\right)$$

$$\nu = ty^{-1}g^{1/2}k^{-1/2}, \qquad \sigma_1 = \sigma k^{-1}, \qquad \omega_1 = \omega(gk)^{-1/2}$$

In an analogous manner we obtain

$$\zeta_3 = \frac{1}{g}\left(\frac{\partial\varphi_3}{\partial t}\right)_{z=0} = \zeta_{31} + \zeta_{32} \tag{1.21}$$

where

$$\zeta_{31} = -\frac{c\sqrt{\sigma^2 - k^2}}{2\pi\sqrt{gk}} e^{ikx} P\int_0^\infty \frac{\omega_1 - \sqrt[4]{1+n^2}}{\sigma_1^2 - n^2 - 1}\left[e^{-kyM_1(n)} + e^{-kyM_2(n)}\right] dn \tag{1.22}$$

$$\zeta_{32} = \frac{ik\sqrt{gk}}{2g} e^{ikx}\int_0^\infty \left(\omega_1 - \sqrt[4]{1+n^2}\right)\int_0^\infty \frac{\mu A(\mu)\sqrt{\mu^2 + k^2}}{k^2 n^2 + \mu^2 + k^2} d\mu \left[e^{-kyM_1(n)} + e^{-kyM_4(n)}\right] dn \tag{1.23}$$

Thus, the equation of the free surface of the fluid at any time $t > 0$ has the form

$$\zeta = \zeta_1 + \zeta_2 + \zeta_3$$

where ζ_1, ζ_2, ζ_3 are represented respectively by Formulas (1.12), (1.18) and (1.21).

2. We shall pass on to the investigation of this solution. Applying the method of stationary phase for large values of ky in investigating the integrals which enter into Expression (1.19), we see that the integrand of the second integral has no stationary points on the positive real axis since $M_2'(n)$ does not vanish here and the Re $M_2(n) \leqslant 0$ on the contour L_1 which consists of the positive real axis with a circuit about the pole $n = n^0 = \sqrt{(\sigma_1^2 - 1)}$ along a semi-circle lying in the lower half-plane. Therefore

$$P\int_0^\infty f(n)\, e^{kyM_1(n)}dn = \int_{(L_1)} f(n)\, e^{kyM_1(n)}dn + \frac{i\pi k}{\sqrt{(\sigma^2-k^2)\,gk}}\, e^{-i(y\sqrt{\sigma^2-k^2}+\omega t)} \qquad (2.1)$$

where

$$f(n) = \frac{\omega_1 + \sqrt[4]{1+n^2}}{\sigma_1^2 - n^2 - 1}$$

We shall introduce the designations

$$\nu_0 = \sqrt[4]{108}, \qquad \nu^0 = \frac{2\omega_1^3}{\sqrt{\omega_1^4 - 1}}$$

It is easily seen that the equation $M_1' = 0$ has no positive roots for $\nu < \nu_0$ and that it has two different positive roots for $\nu > \nu_0$; moreover, Re $M_1(n) \leqslant 0$ for $\nu < \nu^0$ on the contour L_2, which consists of the positive real axis with a circuit about the pole along the upper semi-circle, and also for $\nu > \nu^0$ on the contour L_1. Carrying out the calculation of the residue taking into account the direction of the circuit about the pole, we obtain the following value for the first integral of Formula (1.19):

$$P\int_0^\infty f(n)\, e^{kyM_1(n)}dn = \begin{cases} J + J_3(L_2) & (\nu < \nu^0) \\ J + J_3(L_1) & (\nu > \nu^0) \end{cases} \qquad (2.2)$$

where

$$J = \frac{i\pi k}{\sqrt{(\sigma^2-k^2)\,gk}}\, e^{i(y\sqrt{\sigma^2-k^2}-\omega t)}, \quad J_3(L) = \int_{(L)} f(n)\, e^{kyM_1(n)}dn \qquad (2.3)$$

Making use of the Kelvin method (cf., for example, [3]) to estimate integrals of the form (2.3) for large values of ky, we obtain from Formulas (1.19), (2.1) and (2.2) the following value for ζ_{21}:

$$\zeta_{21} = \begin{cases} icg^{-1}\omega \exp\left[i\left(kx + y\sqrt{\sigma^2-k^2} - \omega t\right)\right] + J_4 & (\nu < \nu^0) \\ J_4 & (\nu > \nu^0) \end{cases} \qquad (2.4)$$

where

$$J_4 = O\left[(ky)^{-1}\right] \text{ for } \nu < \nu_0, \quad J_4 = O\left[(ky)^{-1/2}\right] \text{ for } \nu > \nu_0$$

We shall pass on to the investigation of Formula (1.20). The integrand in this formula as a function of the variable n has no singularities on the real axis; therefore, using the results of the investigation of the expressions $M_1(n)$ and $M_2(n)$ which we carried out, we find the asymptotic value of ζ_{22} for large values of ky

$$\zeta_{22} = O\left[(ky)^{-1}\right] \text{ for } \nu < \nu_0, \qquad \zeta_{22} = O\left[(ky)^{-1/2}\right] \text{ for } \nu > \nu_0 \qquad (2.5)$$

Passing on to the investigation of the expression ζ_3, we note that the essential difference of Expression (1.22) from the integral terms summed in Formula (1.19) which we investigated is that the integrand in (1.22) has no poles on the real axis, and therefore the symbol of the principal value of the integral loses its meaning here and the integration is carried out along the positive real axis. Using the Kelvin method to estimate Expression (1.22) for large values of ky, we have

$$\zeta_{31} = O\,[(ky)^{-1}] \quad \text{for } \nu < \nu_0, \qquad \zeta_{31} = O\,[(ky)^{-1/2}] \text{ for } \nu > \nu_0 \tag{2.6}$$

The investigation of the expression ζ_{32} is carried out in a manner exactly analogous to the investigation of ζ_{22} and leads to the following result:

$$\zeta_{32} = O\,[(ky)^{-1}] \quad \text{for } \nu < \nu_0, \qquad \zeta_{32} = O\,[(ky)^{-1/2}] \quad \text{for } \nu > \nu_0 \tag{2.7}$$

We shall denote by y_1 that value of y beginning with which our asymptotic formula is valid, and we shall introduce the additional designations

$$v_1 = \sqrt{\frac{g}{2k\sqrt{27}}}, \quad v_2 = \frac{g\sqrt{\omega^4 - k^2g^2}}{2\omega^3}, \quad t_1 = \frac{y_1}{v_1}, \quad t_2 = \frac{y_1}{v_2}$$

Then Formulas (1.12), (2.4) to (2.7) give the following expressions for the elevation of the fluid in the region $y > y_1$:

1) Waves of form

$$\zeta = O\,[(ky)^{-1} \qquad \begin{cases} \text{for } t < t_1 \text{ in the region } y > y_1 \\ \text{for } t > t_1 \text{ in the region } y > v_1 t \end{cases} \tag{2.8}$$

2) Waves of form

$$\zeta = O\,[(ky)^{-1/2}] \qquad \begin{cases} \text{for } t_2 > t > t_1 \text{ in the region } y_1 < y < v_1 t \\ \text{for } t > t_2 \quad \text{in the region } v_2 t < y < v_1 t \end{cases} \tag{2.9}$$

3) Waves of form

$$\zeta = \frac{2a\omega^2}{g\sqrt{\sigma^2 - k^2}}\,(1 - e^{-\sigma h}) \sin\,(kx + y\sqrt{\sigma^2 - k^2} - \omega t) \quad \begin{matrix} \text{for } t > t_2 \text{ in} \\ \text{the region } y_1 < y < v_2 t \end{matrix} \tag{2.10}$$

The results of the investigation that has been carried out, which are expressed by Formulas (2.8) to (2.10), indicate that in the case under consideration waves whose amplitudes decrease with increasing distance from the vibrating strip are propagated on the surface of the fluid in the region $y \geqslant y_1$ for time $t < t_2$. For time $t > t_2$ the following picture of the motion of the surface waves arises: in the region $y_1 < y < v_2 t$ steady plane progressive waves of form (2.10) are generated whose fronts move along the y-axis with velocity v_2 equal to the projection on the

y-axis of the group velocity of the waves; in the region $v_2 t < y < v_1 t$ damped waves of form (2.9) are propagated, whose fronts move with velocity v_1 and whose rear boundaries move with velocity v_2 in the positive direction of the y-axis; the region $y > v_1 t$ is filled with damped waves of form (2.8), whose rear boundaries move along the y-axis with velocity v_1.

3. We shall find an expression for the work which is performed by the strip vibrating in the fluid. The work of the strip per wavelength of the strip at time t is represented in the form of a sum of three terms

$$W = W_1 + W_2 + W_3$$

where

$$W_1 = \int_{-h}^{0} dz \int_{x}^{x+2\pi k^{-1}} \rho \left[\left(\frac{\partial \varphi_1}{\partial t}\right)_{y=0} - gz + \frac{p_0}{\rho}\right] a\omega \sin(kx - \omega t)\, dx \qquad (3.1)$$

$$W_j = \int_{-h}^{0} dz \int_{x}^{x+2\pi k^{-1}} \rho \left(\frac{\partial \varphi_j}{\partial t}\right)_{y=0} a\omega \sin(kx - \omega t)\, dx \qquad (j = 2, 3) \qquad (3.2)$$

Since

$$\left(\frac{\partial \varphi_2}{\partial t}\right)_{y=0} = -i \int_{0}^{\infty} B(m)\, \vartheta(m) \exp\left[ikx + Q(m)t + z\sqrt{k^2 + m^2}\right] dm$$

where $Q(m) = -i\vartheta(m)$, thus after substituting in place of $B(m)$ its value given by Formulas (1.16) and (1.17) and after taking into account that $\operatorname{Re} Q(m) \leqslant 0$ on the contour L_1, we obtain

$$W_2 = ic\sqrt{\sigma^2 - k^2} \int_{(L_1)} \frac{\psi(m)}{\sigma^2 - k^2 - m^2} e^{tQ(m)} dm + \\ + \int_{0}^{\infty} \psi(m) \int_{0}^{\infty} \frac{\mu A(\mu)\sqrt{\mu^2 + k^2}}{\mu^2 + k^2 + m^2} d\mu\, e^{tQ(m)} dm \qquad (3.3)$$

where

$$\psi(m) = -\frac{\rho a\omega\left[1 - e^{-h\sqrt{k^2 + m^2}}\right]\left[\omega + \vartheta(m)\right]}{k\sqrt{k^2 + m^2}} e^{i\omega t}$$

Evaluating Expression (3.3) and an analogous expression W_3 for large values of the time t, we have

$$W_2 = O(t^{-1/2}), \quad W_3 = O(t^{-1/2})$$

We shall pass on to the calculation of W_1. Since

$$\left(\frac{\partial \varphi_1}{\partial t}\right)_{y=0} = -i\omega e^{i(kx-\omega t)}\left[ce^{\sigma z} + \int_0^\infty A(\mu)(\mu\cos\mu z + \sigma\sin\mu z)\,d\mu\right]$$

thus after substituting this expression in Formula (3.1) we find

$$W_1 = \frac{\pi c a \omega^2 \rho}{\sigma k}(1 - e^{-\sigma h})$$

Hence we obtain that the work E performed by the strip per wavelength of the strip during a period of vibration of the strip has, with accuracy to quantities of order $t^{-1/2}$, the form

$$E = \int_t^{t+2\pi\omega^{-1}} W_1 dt = \frac{\pi^2 c^2 \rho \sqrt{\sigma^2 - k^2}}{\sigma k} \tag{3.4}$$

We shall now calculate the energy which is carried away by the steady waves of form (2.10) which we shall write as

$$\zeta = N\sin(\sigma x' - \omega t); \qquad \cos\alpha = \frac{\sqrt{\omega^4 - k^2 g^2}}{\omega^2}, \qquad N = \frac{c\omega}{g}$$

where the direction of x' makes the angle a with the y-axis.

The energy E_1 carried by these waves during the time period $2\pi\omega^{-1}$ through a plane of width $2\pi k^{-1}\cos a$ perpendicular to the direction of propagation of the waves is expressed as

$$E_1 = \frac{\pi N^2 \rho g \bar{\lambda} \cos\alpha}{2} = \frac{\pi^2 c^2 \rho \sqrt{\sigma^2 - k^2}}{\sigma k}. \tag{3.5}$$

Comparing Formulas (3.4) and (3.5), we arrive at the conclusion that for large values of the time t the energy of the strip goes completely to the generation of undamped waves.

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